

The time optimal control with constraints of the rectangular type for linear time-varying ODEs

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Abstract

In this paper, we study a time optimal control problem of some linear time-varying ordinary differential equations, where the control constrained set is of the rectangular type. We aim to build up a necessary and sufficient condition and provide an algorithm for the optimal time, as well as the optimal control. We first set up a norm optimal control problem associated with the control constraints of the rectangular type; then establish an equivalence theorem between the time optimal control problem and the aforementioned norm optimal control problem; finally, reach the aim, through utilizing the equivalence theorem and analyzing the variational characterization for the norm optimal control problem.

Key words. time optimal control, norm optimal control, optimal time, optimal norm, control constraints of the rectangular type

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1 Introduction

Let m and d be two natural numbers. Let $A(\cdot) \in C([0, +\infty); \mathbb{R}^{m \times m})$ and $b_i \in \mathbb{R}^m$ with $i = 1, \dots, d$. Consider the following controlled linear time-varying ordinary differential equation:

$$y'(t) + A(t)y(t) = \sum_{i=1}^d b_i u^i(t), \quad t \geq 0, \quad y(0) = y_0. \quad (1.1)$$

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Here and throughout this paper, the initial state y_0 is assumed to be a nonzero vector in \mathbb{R}^m , $u^i(\cdot)$, $i = 1, \dots, d$, are control functions from \mathbb{R}^+ to \mathbb{R}^1 . The following notations will be frequently used in this paper: we denote by $y(\cdot; u)$ the solution of Equation (1.1) corresponding to the control $u = (u^1, \dots, u^d)$; write $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ for the usual inner product and Euclid norm in \mathbb{R}^m respectively; use A^* and $\|A\|_{\mathbb{R}^m \times m}$ to denote the transpose matrix and matrix norm of A respectively. The following assumptions on $A(\cdot)$ will be effective throughout the paper:

(H.1) $A(\cdot)$ is real analytic on $(0, +\infty)$.

(H.2) For each $i = 1, \dots, d$, $(A(\cdot), b_i)$ satisfies Conti's condition, namely, the equality:

$$\int_0^{+\infty} |\langle b_i, \varphi(t) \rangle| dt = +\infty$$

holds for each nonzero solution to the dual equation:

$$\varphi'(t) - A^*(t)\varphi(t) = 0, \quad t \in [0, +\infty). \quad (1.2)$$

With respect to Conti's condition, we refer the readers to [2] or [13]. It is worth to mention that Conti's condition holds for $A(\cdot)$ and $b_1 \in \mathbb{R}^m$ if and only if the system (1.1) where $d = 1$ is null controllable with the control constraints $|u^1(t)| \leq 1$, for a.e. $t > 0$.

Next, we introduce the time optimal control problem with control constraints of the rectangular type. Arbitrarily fix a sequence of numbers $\{k_i\}_{i=1}^d$ such that $1 = k_1 \geq k_2 \geq \dots \geq k_d > 0$. For each $M > 0$, we define the following set:

$$\begin{aligned} \mathcal{U}^M &\triangleq \{u = (u^1, \dots, u^d) : (0, +\infty) \rightarrow \mathbb{R}^d \text{ is measurable;} \\ &|u^i(t)| \leq k_i M, \text{ for a.e. } t \in (0, +\infty) \text{ and for all } i = 1, \dots, d\}. \end{aligned}$$

This set is called a control constrained set with the rectangular type. Now, we introduce the following time optimal control problem:

$$(TP)^M \quad \inf \{T > 0; y(T; u) = 0, u \in \mathcal{U}^M\}.$$

In this problem, the number

$$t^*(M) \triangleq \inf \{T; y(T; u) = 0, u \in \mathcal{U}^M\}$$

is called the optimal time; a control $u_* \in \mathcal{U}^M$, such that $y(t^*(M); u_*) = 0$, is called a time optimal control (or an optimal control, for simplicity); and a control $u \in \mathcal{U}^M$, such that $y(T; u) = 0$ for some $T > 0$, is called an admissible control.

The existence of time optimal controls to $(TP)^M$ has been studied in [13]. In this paper, we build up a necessary and sufficient condition and provide an algorithm for the

optimal time, as well as the optimal control to $(TP)^M$. To present the first main result, we introduce a functional J^T , for each $T > 0$ and each $\{k_i\}_{i=1}^d$, with $1 = k_1 \geq k_2 \geq \dots \geq k_d > 0$, by setting

$$J^T(\varphi_T) = \frac{1}{2} \left(\int_0^T \sum_{i=1}^d k_i |\langle b_i, \varphi(t) \rangle| dt \right)^2 + \langle \varphi(0), y_0 \rangle, \quad \varphi_T \in \mathbb{R}^m, \quad (1.3)$$

where $\varphi(\cdot)$ is the solution to Equation (1.2) with $\varphi(T) = \varphi_T$. It is proved that this functional has a nonzero minimizer in \mathbb{R}^m . Then, the first main result is stated as follows:

Theorem 1.1. *Let $M > 0$. Then, t^* and u_* are the optimal time and the optimal control to $(TP)^M$, respectively, if and only if $t^* > 0$ and $u_* \in \mathcal{U}^M$ satisfy that*

$$u_*^i(t) = k_i M \frac{\langle b_i, \hat{\varphi}(t) \rangle}{|\langle b_i, \hat{\varphi}(t) \rangle|} \quad \text{for a.e. } t \in (0, t^*) \quad \text{and for all } i = 1, \dots, d \quad (1.4)$$

and

$$M = \int_0^{t^*} \sum_{j=1}^d k_j |\langle b_j, \hat{\varphi}(t) \rangle| dt, \quad (1.5)$$

where $\hat{\varphi}(\cdot)$ is the solution to Equation (1.2) with $\varphi(t^*) = \hat{\varphi}_{t^*}$, which is a minimizer of the functional J^{t^*} .

To state the second main result, we define, for each $T > 0$ and each $\{k_i\}_{i=1}^d$, with $1 = k_1 \geq k_2 \geq \dots \geq k_d > 0$, a set of controls:

$$\mathcal{V}^T \triangleq \left\{ v = (v^1, \dots, v^d) \in L^\infty(0, T; \mathbb{R}^d); \quad |v^i(t)| \leq k_i \|v^1\|_{L^\infty(0, T; \mathbb{R})} \right. \\ \left. \text{for a.e. } t \in (0, T) \quad \text{and for all } i = 1, \dots, d \right\};$$

and then introduce the following norm optimal control problem:

$$(NP)^T \quad \inf \left\{ \|v^1\|_{L^\infty(0, T; \mathbb{R})}; y(T; v) = 0, v \in \mathcal{V}^T \right\},$$

where $y(\cdot; v)$ is the solution to Equation (1.1), where the time horizon \mathbb{R}^+ is replaced by $(0, T)$, corresponding to the control $v(\cdot)$. Write

$$\widetilde{M}(T) \triangleq \inf \left\{ \|v^1\|_{L^\infty(0, T; \mathbb{R})}; y(T; v) = 0, v \in \mathcal{V}^T \right\}.$$

Then, we construct a sequence of numbers $\{t_n\}_{n=0}^{+\infty}$ as follows: Let $t_0 > 0$ be arbitrarily given. Let $K \in \mathbb{N}$ be such that

$$K = \min \left\{ k \in \mathbb{N}; \quad \widetilde{M}(kt_0) < M \right\}.$$

It is proved that such a K exists. Then we set $a_0 = 0$, $b_0 = Kt_0$. Write $t_1 = (a_0 + b_0)/2$. In general, when $t_n = (a_{n-1} + b_{n-1})/2$, $n \geq 1$, with a_{n-1} and b_{n-1} being given, it is defined that

$$[a_n, b_n] = \begin{cases} [t_n, b_{n-1}], & \text{if } \widetilde{M}(t_n) > M, \\ [a_{n-1}, t_n], & \text{if } \widetilde{M}(t_n) \leq M \end{cases}$$

and $t_{n+1} = (a_n + b_n)/2$. It is proved that the sequence $\{t_n\}_{n=0}^{+\infty}$ can be determined by solving a series of problems of calculus of variation $\min_{\varphi_T \in \mathbb{R}^m} J^T(\varphi_T)$ with different T . Now, the second main result is presented as follows:

Theorem 1.2. *Suppose that $M > 0$. Let $\{t_n\}_{n=0}^{+\infty}$ be the above-mentioned sequence. Write u_* for the optimal control to $(TP)^M$. Let $u_n = (u_n^1, \dots, u_n^d)$, $n \in \mathbb{N}$, be defined by*

$$u_n^i(t) = \left(\int_0^{t_n} \sum_{j=1}^d k_j |\langle b_j, \hat{\varphi}_n(t) \rangle| dt \right) \frac{k_i \langle b_i, \hat{\varphi}_n(t) \rangle}{|\langle b_i, \hat{\varphi}_n(t) \rangle|} \text{ for a.e. } t \in (0, +\infty), \quad i = 1, \dots, d,$$

where $\hat{\varphi}_n(\cdot)$ is the solution to Equation (1.2) with $\varphi(t_n) = \hat{\varphi}_{t_n}$, which is a minimizer of the functional J^{t_n} . Then, it holds that

$$t_n \rightarrow t^*(M) \text{ as } n \rightarrow +\infty \quad (1.6)$$

and for each i with $1 \leq i \leq d$,

$$u_n^i \rightarrow u_*^i \text{ in } L^2(0, t^*(M); \mathbb{R}). \quad (1.7)$$

The main idea to prove the above theorems is as follows: We first build up an equivalence theorem of our time optimal control problem and the norm optimal control problem constructed above; then make use of the variational characterization of the norm optimal control and the equivalence theorem to show the above two theorems. The aforementioned equivalence theorem is motivated by the analogous equivalence results established for heat equations with control constraints of the ball type in [16] (see also [15]). However, the time optimal control problems with control constraints of the rectangular type differ from those with control constraints of the ball type, from different points of view (see for instance [9]). The equivalence theorem, as well as the structure of the norm optimal control problems in this paper seems to be new.

There have been a lots of literatures on time optimal control problems of differential equations (see, for instance, [3], [5], [8], [7], [9], [10], [11], [12], [13], [14]). Recently, the semi-smooth Newton methods to analyze numerically the time optimal controls with constraints of the cubic type for some ordinary differential equations have been introduced in [6].

To our best knowledge, the necessary and sufficient condition and the algorithm for the optimal time, as well as optimal control, provided in this paper, have not been studied. The equality (1.5) provides a formula for the optimal time to $(TP)^M$.

The rest of this paper is organized as follows: Section 2 provides some results related to the norm optimal control problem $(NP)^T$. Section 3 establishes an equivalence theorem between the norm and the time optimal controls. Section 4 presents the proof of the main theorems.

2 Some Properties about $(NP)^T$

We first present the following properties for the functional J^T which is defined by (1.3).

Lemma 2.1. *For each $T > 0$, J^T is continuous, convex, and coercive in \mathbb{R}^m . Moreover, each minimizer of J^T is nonzero.*

Proof. We first show the existence of minimizers for J^T . The proof of the continuity and convexity of J^T follows from the standard argument (see, for instance, [17], [18]). Now, we show the coercivity of J^T . For this purpose, we set, for each $\varphi_T \in \mathbb{R}^m$,

$$\|\varphi_T\|_* \triangleq \int_0^T \sum_{i=1}^d k_i |\langle b_i, \varphi(t) \rangle| dt, \quad (2.1)$$

where $\varphi(\cdot)$ is the solution to Equation (1.2) with $\varphi(T) = \varphi_T$. Because of (H.1) and (H.2), $\|\cdot\|_*$ is a norm in \mathbb{R}^m . By the equivalence of norms in \mathbb{R}^m , there exists a constant $\Lambda > 0$ such that

$$\|\varphi_T\| \leq \Lambda \|\varphi_T\|_*.$$

This, together with the definition of J^T and (2.1), leads to

$$J^T(\varphi_T) \geq \frac{1}{2\Lambda^2} \|\varphi_T\|^2 - \|y_0\| \|\varphi(0)\|.$$

Thus

$$\lim_{\|\varphi_T\| \rightarrow +\infty} J^T(\varphi_T) = +\infty.$$

Hence, $J^T(\cdot)$ is coercive in \mathbb{R}^m . Therefore, J^T has minimizers in \mathbb{R}^m .

We next show that any minimizer of J^T is nonzero. For this purpose, we set, for each $\alpha > 0$, $\varphi_T^\alpha \triangleq -\alpha \Psi(T, 0) y_0$, where $\Psi(\cdot, \cdot)$ is the fundamental solution associated to

Equation (1.2). Write $\varphi^\alpha(\cdot)$ for the solution to Equation (1.2) with $\varphi(T) = \varphi_T^\alpha$. Then

$$\begin{aligned} J^T(\varphi_T^\alpha) &= \frac{\alpha^2}{2} \left(\int_0^T \sum_{i=1}^d k_i |\langle b_i, \Psi(t, 0)y_0 \rangle| dt \right)^2 - \alpha \|y_0\|^2 \\ &\leq \frac{\alpha^2 T^2}{2} \|\Psi(\cdot, 0)\|_{L^\infty(0, T; \mathbb{R}^{m \times m})}^2 \left(\sum_{i=1}^d k_i \|b_i\| \right)^2 \|y_0\|^2 - \alpha \|y_0\|^2. \end{aligned}$$

This, together with the assumption that $y_0 \neq 0$, implies that $J^T(\varphi_T^\alpha) < 0$ whenever $\alpha > 0$ small enough. Therefore, each minimizer is nonzero. This completes the proof. \square

Remark 2.2. In general, the functional $J^T(\cdot)$ defined by (1.3) is not strictly convex in \mathbb{R}^m . Here we give an example to explain it. Now, assume that $m = 2$, $d = 1$, $T = \pi/4$, $k_1 = 1$ and

$$A(\cdot) \equiv A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then (A, b_1) satisfies Conti's condition. Next, set

$$\varphi_T^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \varphi_T^2 = \begin{pmatrix} 1 + \frac{\sqrt{2}}{2} \\ 1 + \frac{\sqrt{2}}{2} \end{pmatrix}.$$

Let $\varphi^i(\cdot)$ be the solution to Equation (1.2) with $\varphi(T) = \varphi_T^i$, $i = 1, 2$, respectively. Then one can readily check that for each $\lambda \in (0, 1)$, $J^T(\lambda\varphi_T^1 + (1-\lambda)\varphi_T^2) = \lambda J^T(\varphi_T^1) + (1-\lambda)J^T(\varphi_T^2)$. Thus $J^T(\cdot)$ is not strictly convex in \mathbb{R}^2 .

Lemma 2.3. Suppose that $T > 0$. Let $\hat{\varphi}_T$ be a minimizer of J^T . Write $\hat{\varphi}(\cdot)$ for the solution to Equation (1.2) with $\varphi(T) = \hat{\varphi}_T$. Then, the control $\bar{u} = (\bar{u}^1, \dots, \bar{u}^d)$, where

$$\bar{u}^i(t) = \left(\int_0^T \sum_{j=1}^d k_j |\langle b_j, \hat{\varphi}(t) \rangle| dt \right) \frac{k_i \langle b_i, \hat{\varphi}(t) \rangle}{|\langle b_i, \hat{\varphi}(t) \rangle|}, \quad t \in (0, T), \quad i = 1, \dots, d, \quad (2.2)$$

is optimal to $(NP)^T$. Consequently,

$$\widetilde{M}(T) = \int_0^T \sum_{j=1}^d k_j |\langle b_j, \hat{\varphi}(t) \rangle| dt. \quad (2.3)$$

Proof. According to Lemma 2.1, it holds that $\hat{\varphi}_T \neq 0$. This, along with (H.1) and (H.2), indicates that $\langle b_i, \hat{\varphi}(t) \rangle \neq 0$ for a.e. $t \in (0, T)$ and for all $i = 1, \dots, d$.

Next, we derive the Euler-Lagrange equation of the functional J^T associated with $\hat{\varphi}_T$. For each $\varphi_T \in \mathbb{R}^m$, let $\varphi(\cdot)$ be the solution to Equation (1.2) with $\varphi(T) = \varphi_T$. Then

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{1}{h} [J^T(\hat{\varphi}_T + h\varphi_T) - J^T(\hat{\varphi}_T)] &= \langle \varphi(0), y_0 \rangle + \\ \lim_{h \rightarrow 0} \frac{1}{2h} &\left[\left(\int_0^T \sum_{i=1}^d k_i |\langle b_i, \hat{\varphi}(t) + h\varphi(t) \rangle| dt \right)^2 - \left(\int_0^T \sum_{i=1}^d k_i |\langle b_i, \hat{\varphi}(t) \rangle| dt \right)^2 \right] \\ &= \left(\int_0^T \sum_{i=1}^d k_i |\langle b_i, \hat{\varphi}(t) \rangle| dt \right) \int_0^T \sum_{i=1}^d \frac{k_i \langle b_i, \hat{\varphi}(t) \rangle}{|\langle b_i, \hat{\varphi}(t) \rangle|} \langle b_i, \varphi(t) \rangle dt + \langle \varphi(0), y_0 \rangle. \end{aligned}$$

Hence, for each $\varphi_T \in \mathbb{R}^m$, it stands that

$$\left(\int_0^T \sum_{i=1}^d k_i |\langle b_i, \hat{\varphi}(t) \rangle| dt \right) \int_0^T \sum_{i=1}^d \frac{k_i \langle b_i, \hat{\varphi}(t) \rangle}{|\langle b_i, \hat{\varphi}(t) \rangle|} \langle b_i, \varphi(t) \rangle dt + \langle \varphi(0), y_0 \rangle = 0. \quad (2.4)$$

The remainder is to show that \bar{u} is an optimal control to $(NP)^T$. For this purpose, we first observe that for each $v \in \mathcal{V}^T$, $y(T; v) = 0$ if and only if

$$\int_0^T \sum_{i=1}^d v^i(t) \langle b_i, \varphi(t) \rangle dt + \langle \varphi(0), y_0 \rangle = 0, \quad \text{for each } \varphi_T \in \mathbb{R}^m. \quad (2.5)$$

On the other hand, it follows from (2.2) that $\bar{u} \in \mathcal{V}^T$, while it follows from (2.2), (2.4) and (2.5) that $y(T; \bar{u}) = 0$. Now, by taking $\varphi_T = \hat{\varphi}_T$ in both (2.4) and (2.5) respectively, and then using (2.2), we deduce that

$$\int_0^T \sum_{i=1}^d \bar{u}^i(t) \langle b_i, \hat{\varphi}(t) \rangle dt = \int_0^T \sum_{i=1}^d v^i(t) \langle b_i, \hat{\varphi}(t) \rangle dt. \quad (2.6)$$

By making use of (2.2) again, we see that

$$\int_0^T \sum_{i=1}^d \bar{u}^i(t) \langle b_i, \hat{\varphi}(t) \rangle dt = \left(\int_0^T \sum_{i=1}^d k_i |\langle b_i, \hat{\varphi}(t) \rangle| dt \right)^2. \quad (2.7)$$

Since $v \in \mathcal{V}^T$, it stands that

$$\left| \int_0^T \sum_{i=1}^d v^i(t) \langle b_i, \hat{\varphi}(t) \rangle dt \right| \leq \|v^1\|_{L^\infty(0,T;\mathbb{R})} \left(\int_0^T \sum_{i=1}^d k_i |\langle b_i, \hat{\varphi}(t) \rangle| dt \right).$$

This, combined with (2.6) and (2.7), yields that

$$\int_0^T \sum_{i=1}^d k_i |\langle b_i, \hat{\varphi}(t) \rangle| dt \leq \|v^1\|_{L^\infty(0,T;\mathbb{R})},$$

from which and (2.2), it follows that

$$\|\bar{u}^1\|_{L^\infty(0,T;\mathbb{R})} \leq \|v^1\|_{L^\infty(0,T;\mathbb{R})}.$$

This completes the proof. □

3 Equivalence of Time and Norm Optimal Controls

The main purpose of this section is to show the following equivalence theorem:

Theorem 3.1. *For each $T > 0$, the norm optimal control to $(NP)^T$, when is extended to $(0, +\infty)$ by taking zero value on $(T, +\infty)$, is the time optimal control to $(TP)^{\widetilde{M}(T)}$. On the other hand, for each $M > 0$, the time optimal control to $(TP)^M$, when is restricted over $(0, t^*(M))$, is the norm optimal control to $(NP)^{t^*(M)}$.*

We start with introducing three lemmas as follows:

Lemma 3.2. *For each $M > 0$, $(TP)^M$ has a unique optimal control over $(0, t^*(M))$. Furthermore, it has the strong bang-bang property: any optimal control $u_* = (u_*^1, \dots, u_*^d)$ satisfies that $|u_*^i(t)| = k_i M$, for a.e. $t \in (0, t^*(M))$ and for all $i = 1, \dots, d$.*

Proof. Since (H.2) stands, there exists a control $u \in \mathcal{U}^M$ such that $y(T; u) = 0$ for some $T > 0$ (see [2], [13]). By a standard argument (see, for instance, [7]), the existence of time optimal controls to $(TP)^M$ follows immediately.

Next, let u_* be an optimal control to $(TP)^M$. By the Pontryagin maximum principle (see, for instance, [11]), there exists a nonzero solution $\varphi(\cdot)$ to Equation (1.2) such that

$$\sum_{i=1}^d \max_{|v^i| \leq k_i M} \langle b_i, \varphi(t) \rangle v^i = \sum_{i=1}^d \langle b_i, \varphi(t) \rangle u_*^i \text{ for a.e. } t \in (0, t^*(M)). \quad (3.1)$$

Because of (H.1) and (H.2), it holds that $\langle b_i, \varphi(t) \rangle \neq 0$ for a.e. $t \in (0, t^*(M))$ and for all $i = 1, \dots, d$. This, together with (3.1), yields that

$$u_*^i(t) = k_i M \frac{\langle b_i, \varphi(t) \rangle}{|\langle b_i, \varphi(t) \rangle|} \text{ for a.e. } t \in (0, t^*(M)) \text{ and for all } i = 1, \dots, d.$$

Hence, the desired strong bang-bang property follows immediately. Finally, by the strong bang-bang property, the uniqueness of the time optimal control over $(0, t^*(M))$ follows from the standard argument (see, for instance, [5]). \square

Lemma 3.3. *Let $T > 0$. For each $\tau \in [0, T)$ and each $z_0 \in \mathbb{R}^m$, there exists a control $u^1 \in L^\infty(\tau, T; \mathbb{R})$ such that the solution $y(\cdot; u^1)$ to the following equation:*

$$y'(t) + A(t)y(t) = b_1 u^1(t), \quad y(\tau) = z_0, \quad (3.2)$$

verifies $y(T; u^1) = 0$. Moreover, the control u^1 satisfies the following estimate:

$$\|u^1\|_{L^\infty(\tau, T; \mathbb{R})} \leq C \|z_0\|,$$

where C is a positive constant independent of z_0 .

Proof. Since (H.1) stands and $(A(\cdot), b_1)$ satisfies Conti's condition (see (H.2)), the system (3.2) holds the unique continuation property on (τ, T) . Then, applying the Theorem 5 in Chapter 3 in [12], we get that the controllability Gramian $W(\tau, T)$ is positive definite, where

$$W(\tau, T) = \int_{\tau}^T \Phi(T, s) b_1 b_1^* \Phi^*(T, s) ds.$$

Here, $\Phi(\cdot, \cdot)$ is the fundamental solution associated to $A(\cdot)$. Next, set

$$u^1(t) = -b_1^* \Phi^*(T, t) W(\tau, T)^{-1} \Phi(T, \tau) z_0, \quad t \in [\tau, T). \quad (3.3)$$

It can be easily checked that $y(T; u^1) = 0$. By (3.3), it holds that

$$\|u^1\|_{L^\infty(\tau, T; \mathbb{R})} \leq \|b_1^* \Phi^*(T, \cdot)\|_{L^\infty(\tau, T; \mathbb{R}^m)} \|W(\tau, T)^{-1}\|_{\mathbb{R}^m \times m} \|\Phi(T, \tau)\|_{\mathbb{R}^m \times m} \|z_0\|.$$

Hence, there exists a constant $C > 0$ (independent of z_0) such that

$$\|u^1\|_{L^\infty(0, T; \mathbb{R})} \leq C \|z_0\|.$$

This completes the proof. \square

Next lemma concerns some properties of the map $M \rightarrow t^*(M)$.

Lemma 3.4. *The optimal time function $t^*(\cdot)$ is strictly monotonically decreasing and continuous. In addition, it holds that $\lim_{M \rightarrow +\infty} t^*(M) = 0$ and $\lim_{M \rightarrow 0^+} t^*(M) = +\infty$.*

Proof. We carry out the proof by five steps as follows:

Step 1: The function $t^(\cdot)$ is strictly monotonically decreasing.*

Let $M_1 > M_2 > 0$. It suffices to show that $t^*(M_1) < t^*(M_2)$. To this end, let u_2 be the optimal control to $(TP)^{M_2}$. Clearly, u_2 is admissible for $(TP)^{M_1}$. By the optimality of $t^*(M_1)$ to $(TP)^{M_1}$, it is clear that $t^*(M_1) \leq t^*(M_2)$. Next, suppose by contradiction that $t^*(M_1) = t^*(M_2)$. Then, it would hold that

$$y(t^*(M_1); u_2) = y(t^*(M_2); u_2) = 0.$$

Hence, u_2 is also the optimal control to $(TP)^{M_1}$. By Lemma 3.2, we find that

$$k_i M_1 = |u_2^i(t)| = k_i M_2 \text{ for a.e. } t \in (0, t^*(M_1)) \text{ and for all } i = 1, \dots, d.$$

This leads to a contradiction, since $M_2 < M_1$. Therefore, it holds that $t^*(M_1) < t^*(M_2)$.

Step 2: The function $t^(\cdot)$ is continuous from right, that is, $\lim_{M_n \searrow M} t^*(M_n) = t^*(M)$.*

Let $M_1 > M_2 > \dots > M_n > \dots > M > 0$ and $\lim_{n \rightarrow +\infty} M_n = M$. By Step 1, it holds that

$$t^*(M_1) < t^*(M_2) < \dots < t^*(M_n) < \dots < t^*(M) \quad \text{and} \quad \lim_{n \rightarrow +\infty} t^*(M_n) \leq t^*(M).$$

We claim that $\lim_{n \rightarrow +\infty} t^*(M_n) = t^*(M)$. Seeking a contradiction, we suppose that

$$\lim_{n \rightarrow +\infty} t^*(M_n) = t^*(M) - \delta \text{ for some } \delta > 0.$$

Clearly, the optimal controls u_n to $(TP)^{M_n}$, $n \in \mathbb{N}$, satisfy that

$$\|u_n^i\|_{L^\infty(\mathbb{R}^+; \mathbb{R})} \leq k_i M_n < k_i M_1, \text{ for all } i = 1, \dots, d$$

and

$$y(t^*(M_n); u_n) = 0.$$

Thus, on a subsequence, $u_n \rightarrow \tilde{u}$ weakly star in $L^\infty(\mathbb{R}^+; \mathbb{R}^d)$. Furthermore, one can easily derive from the above observations that $\tilde{u} \in \mathcal{U}^M$ and $y(t^*(M) - \delta; \tilde{u}) = 0$. These contradict with the optimality of $t^*(M)$ to $(TP)^M$.

Step 3: The function $t^(\cdot)$ is continuous from left, that is, $\lim_{M_n \nearrow M} t^*(M_n) = t^*(M)$.*

Let $0 < M_1 < M_2 < \dots < M_n < \dots < M$ and $\lim_{n \rightarrow \infty} M_n = M$. It is clear that

$$t^*(M_1) > t^*(M_2) > \dots > t^*(M_n) > \dots > t^*(M) \quad \text{and} \quad \lim_{n \rightarrow +\infty} t^*(M_n) \geq t^*(M).$$

Seeking a contradiction, suppose $\lim_{n \rightarrow +\infty} t^*(M_n) > t^*(M)$. Then there would exist a $\delta > 0$ such that

$$\lim_{n \rightarrow +\infty} t^*(M_n) = t^*(M) + \delta.$$

Clearly,

$$t^*(M_n) > t^*(M) + \delta \text{ for all } n \in \mathbb{N}. \quad (3.4)$$

Let u_* be the optimal control to $(TP)^M$. Set $\delta_n = \frac{M_n}{M}$, $z_n(\cdot) = \delta_n y(\cdot; u_*)$, $n \in \mathbb{N}$. It is clear that

$$\begin{cases} z'_n(t) + A(t)z_n(t) = \sum_{i=1}^d \delta_n b_i u_*^i(t), & t \in (0, t^*(M)), \\ z_n(0) = \delta_n y_0, & z_n(t^*(M)) = 0. \end{cases} \quad (3.5)$$

According to Lemma 3.3, there exist a constant $C > 0$ independent of n and a control f_n^1 with

$$\|f_n^1\|_{L^\infty(t^*(M), t^*(M)+\delta; \mathbb{R})} \leq C \cdot (1 - \delta_n) \|y_0\|,$$

such that

$$\phi_n(t^*(M) + \delta) = 0,$$

where $\phi_n(\cdot)$ solves the equation:

$$\begin{cases} \phi'(t) + A(t)\phi(t) = b_1 f_n^1(t) \chi_{(t^*(M), t^*(M)+\delta)}(t), & t \in (0, t^*(M) + \delta), \\ \phi(0) = (1 - \delta_n) y_0. \end{cases} \quad (3.6)$$

Now, we construct, for each $n \in \mathbb{N}$, a control $g_n = (g_n^1, \dots, g_n^d)$, by setting

$$\begin{cases} g_n^1 = \delta_n u_*^1 \chi_{(0, t^*(M))} + f_n^1 \chi_{(t^*(M), t^*(M)+\delta)}, \\ g_n^i = \delta_n u_*^i \chi_{(0, t^*(M))}, & i = 2, \dots, d. \end{cases} \quad (3.7)$$

Since $\delta_n \nearrow 1$, there exists a positive integer N_1 such that

$$C \cdot (1 - \delta_n) \|y_0\| \leq M_1 < M_n \text{ for all } n \geq N_1.$$

This, along with (3.7), leads to that when $n \geq N_1$

$$\|g_n^i\|_{L^\infty(\mathbb{R}^+; \mathbb{R})} \leq k_i M_n, \text{ for all } i = 1, \dots, d.$$

Finally, set $w_n = z_n + \phi_n$, $n \geq N_1$. It follows at once from (3.5) and (3.6) that

$$\begin{cases} w'_n + A(t)w_n(t) = \sum_{i=1}^d b_i g_n^i(t), & t \in (0, t^*(M) + \delta), \\ w_n(0) = y_0, & w_n(t^*(M) + \delta) = 0. \end{cases}$$

Thus, g_n is admissible to $(TP)^{M_n}$ for each n with $n \geq N_1$. Consequently, $t^*(M_n) \leq t^*(M) + \delta$ whenever $n \geq N_1$. This, together with (3.4), leads to a contradiction.

Step 4: It holds that $\lim_{M \rightarrow 0^+} t^(M) = +\infty$.*

Seeking a contradiction, suppose that there did exist a sequence $\{M_n\}_{n \geq 1}$, with $M_1 > M_2 > \dots > M_n > \dots > 0$ and $\lim_{n \rightarrow \infty} M_n = 0$, such that $\lim_{n \rightarrow \infty} t^*(M_n) = T < +\infty$. Then, the optimal controls u_n to $(TP)^{M_n}$, $n \in \mathbb{N}$, satisfy that on a subsequence, $y(\cdot; u_n) \rightarrow y(\cdot; 0)$ in $C([0, T]; \mathbb{R}^d)$, which leads to a contradiction, since $y_0 \neq 0$.

Step 5: $\lim_{M \rightarrow +\infty} t^(M) = 0$*

Seeking a contradiction, suppose that there existed a $T > 0$ and a sequence $\{M_n\}_{n \geq 1}$, with $0 < M_1 < M_2 < \dots < M_n < \dots$ and $\lim_{n \rightarrow \infty} M_n = +\infty$, such that $\lim_{n \rightarrow +\infty} t^*(M_n) = T$. Let $\delta > 0$ such that $T - \delta > 0$. Then, by Lemma 3.3, there would exist a control u_δ^1 with

$$\|u_\delta^1\|_{L^\infty(0, T-\delta; \mathbb{R})} \leq C\|y_0\|,$$

such that

$$y(T - \delta; (u_\delta^1, \underbrace{0, \dots, 0}_{d-1})) = 0. \quad (3.8)$$

Since $\lim_{n \rightarrow +\infty} M_n = +\infty$, it holds that $C\|y_0\| \leq M_n$ for n large enough. This, together with (3.8), leads to a contradiction to the optimality of $t^*(M_n)$ to $(TP)^{M_n}$.

In summary, we conclude that all statements in this lemma stand. □

Proof of Theorem 3.1. We begin with proving the identity

$$T = t^*(\widetilde{M}(T)) \text{ for each } T > 0. \quad (3.9)$$

From the definition of $\widetilde{M}(T)$ and the optimality of $t^*(\widetilde{M}(T))$ to $(TP)^{\widetilde{M}(T)}$, we can deduce that for each $T > 0$, $t^*(\widetilde{M}(T)) \leq T$. Thus, it suffices to show that the inequality $t^*(\widetilde{M}(T)) < T$ does not stand for each $T > 0$. Suppose by contradiction that $t^*(\widetilde{M}(T)) < T$ for some $T > 0$. Then, by making use of Lemma 3.4, we could find a positive number M_1 , with $M_1 < \widetilde{M}(T)$, such that $t^*(M_1) = T$. It follows from Lemma 3.2 that $(TP)^{M_1}$ has a unique optimal control u_* verifying

$$|u_*^i(t)| = k_i M_1 \text{ for a.e. } t \in (0, T) \text{ and for all } i = 1, \dots, d.$$

Thus, $u_* \in \mathcal{V}^T$ and $y(T; u_*) = 0$. This contradicts with the optimality of $\widetilde{M}(T)$ to $(NP)^T$. Therefore, the equality (3.9) stands.

Now, any optimal control f_* to $(NP)^T$ satisfies that $|f_*^i(t)| \leq k_i \widetilde{M}(T)$ for a.e. $t \in (0, T)$ and all $i = 1, \dots, d$, and $y(T; f_*) = 0$. These, along with (3.9), imply that $f_* \in \mathcal{U}^{\widetilde{M}(T)}$ and $y(t^*(\widetilde{M}(T)); f_*) = 0$. Hence, f_* is the optimal control to $(TP)^{\widetilde{M}(T)}$.

On the other hand, it follows from (3.9) and the strict monotonicity of the function $t^*(\cdot)$ that

$$\widetilde{M}(t^*(M)) = M, \text{ for each } M > 0. \quad (3.10)$$

Thus, the optimal control u_* to $(TP)^M$ is the optimal control to $(TP)^{\widetilde{M}(t^*(M))}$. Then, by the optimality of u_* and by Lemma 3.2, we see that $\|u_*^1\|_{L^\infty(0, t^*(M); \mathbb{R})} = \widetilde{M}(t^*(M))$, $u_* \in \mathcal{V}^{t^*(M)}$ and $y(t^*(M); u_*) = 0$. Hence, u_* is an optimal control to $(NP)^{t^*(M)}$. This completes the proof of Theorem 3.1.

We end this section with the following two consequences.

Corollary 3.5. *For each $T > 0$, $(NP)^T$ has a unique optimal control $f_* = (f_*^1, \dots, f_*^d)$. It is given by*

$$f_*^i(t) = \left(\int_0^T \sum_{j=1}^d k_j |\langle b_j, \hat{\varphi}(t) \rangle| dt \right) \frac{k_i \langle b_i, \hat{\varphi}(t) \rangle}{|\langle b_i, \hat{\varphi}(t) \rangle|}, \text{ for a.e. } t \in (0, T) \text{ and for all } i = 1, \dots, d,$$

where $\hat{\varphi}(\cdot)$ is the solution to Equation (1.2) with $\varphi(T) = \hat{\varphi}_T$, which is a minimizer of the functional J^T . Consequently,

$$\widetilde{M}(T) = \int_0^T \sum_{j=1}^d k_j |\langle b_j, \hat{\varphi}(t) \rangle| dt.$$

Proof. It suffices to show the uniqueness, because of Lemma 2.3. For this purpose, we suppose, by contradiction, that g_* were an optimal control different from f_* . Then, according to Theorem 3.1, both f_* and g_* were optimal controls to $(TP)^{\widetilde{M}(T)}$. By Lemma 3.2, as well as (3.9), they are the same over $(0, T)$, which leads to a contradiction. This completes the proof. \square

Corollary 3.6. *The functions $\widetilde{M}(\cdot)$ and $t^*(\cdot)$ are inverse one of each other. Consequently, $\widetilde{M}(\cdot)$ is strictly monotonically decreasing and continuous. In addition, it holds that*

$$\lim_{T \rightarrow 0^+} \widetilde{M}(T) = +\infty \text{ and } \lim_{T \rightarrow +\infty} \widetilde{M}(T) = 0.$$

Proof. According to Theorem 3.1, it follows that

$$t^* \circ \widetilde{M}(T) = t^*(\widetilde{M}(T)) = T, \text{ for each } T > 0$$

and

$$\widetilde{M} \circ t^*(M) = \widetilde{M}(t^*(M)) = M, \text{ for each } M > 0.$$

Hence, $\widetilde{M}(\cdot)$ is the inverse function of $t^*(\cdot)$. The remainders follow at once from Lemma 3.4. This completes the proof. \square

4 Proof of Theorem 1.1 and 1.2

Proof of Theorem 1.1. Suppose that t^* and u_* are the optimal time and the optimal control to $(TP)^M$ respectively. It is clear that $t^* = t^*(M)$. This, combining with Theorem 3.1, yields that u_* is an optimal control to $(NP)^{t^*}$. According to Corollary 3.5, each $u_*^i(\cdot)$, with $i \in \{1, \dots, d\}$, satisfies

$$u_*^i(t) = \left(\int_0^{t^*} \sum_{j=1}^d k_j |\langle b_j, \hat{\varphi}(t) \rangle| dt \right) \frac{k_i \langle b_i, \hat{\varphi}(t) \rangle}{|\langle b_i, \hat{\varphi}(t) \rangle|} \text{ for a.e. } t \in (0, t^*), \quad (4.1)$$

where $\hat{\varphi}(\cdot)$ is the solution to Equation (1.2) with $\varphi(t^*) = \hat{\varphi}_{t^*}$, which is a minimizer of the functional J^{t^*} . Consequently,

$$\widetilde{M}(t^*) = \int_0^{t^*} \sum_{j=1}^d k_j |\langle b_j, \hat{\varphi}(t) \rangle| dt.$$

This, along with the fact $t^* = t^*(M)$ and the identity (3.10), leads to (1.5). The equality (1.4) follows immediately from (1.5) and (4.1).

Conversely, suppose that $t^* > 0$ and $u_* \in \mathcal{U}^M$ satisfy the equality (1.4) and (1.5). We are going to show that they are the optimal time and the optimal control to $(TP)^M$ respectively. For this purpose, we apply Corollary 3.5 to obtain that u_* is the unique optimal control to $(NP)^{t^*}$. It is clear that $\widetilde{M}(t^*) = M$. By the strict monotonicity of $\widetilde{M}(\cdot)$ (see Corollary 3.6) and the equality (3.10), it holds that $t^* = t^*(M)$. According to Theorem 3.1, u_* is the optimal control to $(TP)^{\widetilde{M}(t^*(M))}$. This, along with (3.10), yields that u_* is the optimal control to $(TP)^M$ and completes the proof of Theorem 1.1.

Remark 4.1. Before giving the proof of Theorem 1.2, we explain the well-posedness of the sequence $\{t_n\}_{n=0}^{+\infty}$ built up in Section 1. In fact, for each $T > 0$, we can determine the

value $\widetilde{M}(T)$ by solving the minimization problem: $\min_{\varphi_T \in \mathbb{R}^m} J^T(\varphi_T)$ (see Corollary 3.5). Since the map $T \rightarrow \widetilde{M}(T)$ is strictly monotonically decreasing and $\widetilde{M}(T)$ tends to 0 as T goes to $+\infty$ (see Corollary 3.6), K can be confirmed by solving a finite number of minimizers of functionals J^T , corresponding to $T = lt_0$, $l = 1, 2, \dots, K$. On the other hand, for each $n \in \mathbb{N}$, t_{n+1} is determined by solving the same minimization problem with $T = t_n$. Hence, the sequence $\{t_n\}_{n=0}^{+\infty}$ can be determined by solving a series of minimizers of functionals J^T with $T = lt_0$, $l = 1, 2, \dots, K$, and $T = t_n$, $n = 1, 2, \dots$.

Proof of Theorem 1.2. We start with proving (1.6). From the structure of the sequence $\{t_n\}_{n=0}^{+\infty}$, it is clear that $t_n \in [a_n, b_n] \subseteq [a_{n-1}, b_{n-1}]$ and $b_n - a_n = (b_{n-1} - a_{n-1})/2$. Hence, it stands that

$$\lim_{n \rightarrow +\infty} t_n = \lim_{n \rightarrow +\infty} a_n = \lim_{n \rightarrow +\infty} b_n. \quad (4.2)$$

Since the function $\widetilde{M}(\cdot)$ is continuous (see Corollary 3.6) and $\widetilde{M}(a_n) > M \geq \widetilde{M}(b_n)$ (which also follows from the structure of $\{t_n\}_{n=0}^{+\infty}$), we see that

$$M = \widetilde{M}\left(\lim_{n \rightarrow +\infty} t_n\right).$$

This, together with (3.10) and the strict monotonicity of the function $\widetilde{M}(\cdot)$, leads to the desired convergence (1.6).

Next, we claim that for each $i = 1, \dots, d$,

$$u_n^i \rightarrow u_*^i \text{ weakly star in } L^\infty(0, t^*(M); \mathbb{R}). \quad (4.3)$$

In fact, for each $n \in \mathbb{N}$, it follows from Corollary 3.5 that u_n , when is restricted over $(0, t_n)$, is the unique optimal control to $(NP)^{t_n}$. Consequently, $y(t_n; u_n) = 0$. We arbitrarily take a subsequence of $\{u_n\}$, denoted by $\{u_{n_k}\}$. Clearly, there exists a subsequence $\{u_{n_j}\}$ of $\{u_{n_k}\}$ such that for each i with $1 \leq i \leq d$,

$$u_{n_j}^i \rightarrow \tilde{u}^i \text{ weakly star in } L^\infty(0, t^*(M); \mathbb{R}). \quad (4.4)$$

Moreover, one can derive from the above facts that $y(t^*(M); \tilde{u}) = 0$. On the other hand, by (1.6) and (3.10), it follows that for each $i = 1, \dots, d$,

$$\|\tilde{u}^i\|_{L^\infty(0, t^*(M); \mathbb{R})} \leq \liminf_{j \rightarrow +\infty} \|u_{n_j}^i\|_{L^\infty(0, t^*(M); \mathbb{R})} = \liminf_{j \rightarrow +\infty} k_i \widetilde{M}(t_{n_j}) = k_i \widetilde{M}(t^*(M)) = k_i M.$$

Hence, \tilde{u} is an optimal control to $(TP)^M$. By the uniqueness of the optimal control to $(TP)^M$, it holds that $\tilde{u} = u_*$ over $(0, t^*(M))$. Therefore, (4.3) follows from (4.4).

Now, we verify the convergence (1.7). By (4.3), we find that for each i with $1 \leq i \leq d$,

$$u_n^i \rightarrow u_*^i \text{ weakly in } L^2(0, t^*(M); \mathbb{R}). \quad (4.5)$$

Clearly, by the strong bang-bang property of $(TP)^M$ (see Lemma 3.2), it follows that

$$|u_*^i(t)| = k_i M, \text{ for a.e. } t \in (0, t^*(M)) \text{ and for all } i = 1, \dots, d.$$

Hence, for each i with $1 \leq i \leq d$, it holds that

$$\|u_n^i\|_{L^2(0, t^*(M); \mathbb{R})} \longrightarrow \|u_*^i\|_{L^2(0, t^*(M); \mathbb{R})} \text{ as } n \rightarrow +\infty.$$

This, along with (4.5), leads to (1.7) and completes the proof of Theorem 1.2.

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References

- [1] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, New York, 2011.
- [2] R. Conti, Teoria del Controllo e del Controllo Ottimo, UTET, Torino, Italy, 1974.
- [3] L. C. Evans, An Introduction to Mathematical Optimal Control Theory, Lecture notes available at <http://math.berkeley.edu/evans/course.pdf>.
- [4] I. Ekeland, R. Temam, Convex Analysis and Variational Problems, North-Holland Mathematics Studies 1, ELSEVIER, 1976.
- [5] H. O. Fattorini, Infinite Dimensional Linear Control Systems: The Time Optimal and Norm Optimal Problems, North-Holland Mathematics Studies 201, ELSEVIER, 2005.
- [6] K. Ito, K. Kunish, Semi-smooth Newton methods for time-optimal control for a class of ODEs, SIAM J. Control Optim., Vol. 48, No.6 (2010), 3997-4013.
- [7] E. B. Lee, L. Markus, Foundations of Optimal Control Theory, John Wiley and Sons, New York, 1967.
- [8] P. Lin, G. Wang, Blowup time optimal control for ordinary differential equations, SIAM J. Control Optim., Vol. 49, No.1 (2011), 73-105.
- [9] Q. Lü, G. Wang, On the existence of time optimal controls with constraints of the rectangular type for heat equations, SIAM J. Control Optim., Vol. 49, No. 3, (2011), 1124-1149.

- [10] V. J. Mizel, T. I. Seidman, An abstract 'bang-bang principle' and time optimal boundary control of the heat equation, SIAM J. Control Optim., Vol. 35, No.4 (1997), 1204-1216.
- [11] L. S. Pontryagin, V. G. Boltyanski, R. V. Gamkrelidze, and E. F. Mishchenko, The Mathematical Theory of Optimal Processes, John Wiley and Son, New York, 1962.
- [12] E. D. Sontag, Mathematical Control Theory: Deterministic Finite Dimensional Systems, Springer, New York, 1998.
- [13] W. E. Schmitendorf, B. R. Barmish, Null controllability of linear system with constrained controls, SIAM J. Control Optim., Vol.18, No.4 (1980), 327-345.
- [14] G. Wang, L^∞ -null controllability for the heat equation and its consequences for the time optimal control problem, SIAM J. Control Optim., 47 (2008), 1701-1720.
- [15] G. Wang and Y. Xu, Equivalence of three different kinds of optimal control problems for heat equations and its applications. <http://arxiv.org/abs/1110.3885>.
- [16] G. Wang, E. Zuazua, On the equivalence between time and norm optimal controls for heat equations, preprint.
- [17] E. Zuazua, Controllability and observability of partial differential equations: Some results and open problems, Handbook of Differential Equations: Evolutionary Differential Equations, Vol.3, Elsevier Science, 2006, 527-621.
- [18] E. Zuazua, Switching control, J. Eur. Math. Soc., Vol. 13 (2011), 85-117.